COROLLARY. EVERY POINT OF ω^* IS A 2^ω-POINT.

DEFINITION. LET $\mathcal{A}$ BE AN INFINITE ALMOST DISJOINT FAMILY CONSISTING OF COUNTABLE SUBSETS OF SOME SET X.

$$J^+(\mathcal{A}) = \{M \subseteq X : |\{A \in \mathcal{A} : |A \cap M| = \omega\}| \geq \omega\}$$

WHAT ARE THE LARGEST FAMILIES, WHICH HAVE AN ALMOST DISJOINT REFINEMENT? WHAT IS THE STRONGEST STATEMENT ABOUT THE EXISTENCE OF AN ALMOST DISJOINT REFINEMENT?

OBSERVATION 13. SUPPOSE $\mathcal{M} \subseteq [X]^{\leq\omega}$ HAS AN ALMOST DISJOINT REFINEMENT. THEN THERE IS AN ALMOST DISJOINT FAMILY $\mathcal{B}$ WITH $\mathcal{M} \subseteq J^+(\mathcal{B})$.

PROOF. SUPPOSE THAT $\mathcal{A}$ IS AN ALMOST DISJOINT REFINEMENT OF $\mathcal{M}$. FOR EACH $A \in \mathcal{A}$, CHOOSE AN INFINITE ALMOST DISJOINT FAMILY $\mathcal{B}(A)$ CONSISTING OF INFINITE SUBSETS OF A. PUT $\mathcal{B} = \bigcup\{\mathcal{B}(A) : A \in \mathcal{A}\}$. WHENEVER $M \supseteq A$, THEN $M \cap B$ IS INFINITE FOR EACH $B \in \mathcal{B}(A)$. SO $\mathcal{M} \subseteq \mathcal{J}^+(\mathcal{B})$. $\square$

FOR A CARDINAL κ, ABBREVIATE:

$RPC(\kappa) \equiv$ "FOR EVERY INFINITE ALMOST DISJOINT FAMILY $\mathcal{A} \subseteq [\kappa]^\omega$, $\mathcal{J}^+(\mathcal{A})$ HAS AN ALMOST DISJOINT REFINEMENT."

THIS IS THE STATEMENT WE LOOKED FOR.

DEFINITION [HECHLER 1971] AN ALMOST DISJOINT FAMILY $\mathcal{A} \subseteq [\kappa]^\omega$ IS CALLED COMPLETELY SEPARABLE, IF $\mathcal{A}$ IS INFINITE AND FOR EACH $M \in \mathcal{J}^+(\mathcal{A})$, THERE IS SOME $A \in \mathcal{A}$ WITH $A \subseteq M$, I.E., $\mathcal{A}$ IS AN ALMOST DISJOINT REFINEMENT OF $\mathcal{J}^+(\mathcal{A})$.

OBSERVATION 14. SUPPOSE $\mathcal{A}$ IS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY. THEN $|\mathcal{A}| \geq 2^\omega$.

PROOF. CHOOSE $\{A_n : n \in \omega\} \subseteq \mathcal{A}$. THIS IS POSSIBLE, SINCE $\mathcal{A}$ IS INFINITE. FOR EACH $n \geq 1$, SPLIT A_n INTO 2^n DISJOINT INFINITE PARTS AND ENUMERATE THEM AS $\{M_s : s \in {}^n2\}$. FOR EACH $f \in {}^\omega 2$,

LET $M_f = \bigcup\{M_{f\restriction n} : 1 \leq n < \omega\}$.
EACH SET M_f BELONGS TO $\mathcal{J}^+(\mathcal{A})$, SO THERE IS SOME $A_f \in \mathcal{A}$ WITH $A_f \subseteq M_f$. THE FAMILY $\{A_f : f \in {}^\omega 2\}$ IS OF SIZE 2^ω. $\square$

OBSERVATION 15. LET $\mathcal{A}$ BE A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY.

(i) IF $M \in \mathcal{J}^+(\mathcal{A})$, THEN BOTH FAMILIES
$\{M \cap A : A \in \mathcal{A}$ AND $|A \cap M| = \omega\}$,
$\{A \in \mathcal{A} : A \subseteq M\}$ ARE COMPLETELY SEPARABLE.

(ii) IF $\mathcal{A}' \subseteq \mathcal{A}$ SATISFIES $|\mathcal{A}'| < 2^\omega$, THEN $\mathcal{A} \setminus \mathcal{A}'$ IS COMPLETELY SEPARABLE.

(iii) IF FOR EACH $A \in \mathcal{A}$, $B(A) \in [A]^\omega$, THE THE FAMILY $\{B(A) : A \in \mathcal{A}\}$ IS COMPLETELY SEPARABLE.

PROOF. TRIVIAL. □

OBSERVATION 16. LET $\mathcal{A}$ BE A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY. THEN FOR EVERY DECREASING SEQUENCE $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots \supseteq X_n \supseteq \dots$ OF SETS FROM $\mathcal{J}^+(\mathcal{A})$, THERE IS SOME $Y \in \mathcal{J}^+(\mathcal{A})$ SATISFYING $Y \subseteq^* X_n$ FOR EACH $n \in \omega$.

PROOF. PROCEED BY INDUCTION. BY COMPLETE SEPARABILITY, CHOOSE FOR EACH $n \in \omega$ A SET $A_n \in \mathcal{A}$ WITH $A_n \subseteq X_n \setminus \bigcup_{i<n} A_i$. PUT $Y = \bigcup_{n<\omega} A_n$. □

OBSERVATION 17. LET $\mathcal{B}$ BE AN INFINITE ALMOST DISJOINT FAMILY SUCH FOR EACH $X \in \mathcal{J}^+(\mathcal{B})$, $|\{B \in \mathcal{B} : |B \cap X| = \omega\}| = 2^\omega$. THEN THERE IS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY $\mathcal{A}$ SATISFYING $\mathcal{J}^+(\mathcal{A}) = \mathcal{J}^+(\mathcal{B})$.

PROOF. FOR EACH $X \in \mathcal{J}^+(\mathcal{B})$ LET $B(X) \in \mathcal{B}$ BE SUCH THAT $B(X) \cap X$ IS INFINITE, $B(X) \neq B(X')$ FOR DISTINCT $X, X' \in \mathcal{J}^+(\mathcal{B})$.
LET $\mathcal{A}$ CONSIST OF ALL $X \cap B(X)$ FOR $X \in \mathcal{J}^+(\mathcal{B})$. $\square$

DEFINITION. A CHAIN OF LENGTH γ IS A FAMILY $\mathcal{T} = \{T_\alpha : \alpha < \gamma\} \subseteq [\omega]^\omega$ SATISFYING $T_\alpha \underset{F}{\overset{*}{\supsetneq}} T_\beta$ WHENEVER $\alpha < \beta < \gamma$.

TWO CHAINS $\mathcal{T}$ AND $\mathcal{T}'$ ARE CALLED **DISJOINT** IF THERE ARE $T \in \mathcal{T}$ AND $T' \in \mathcal{T}'$ WITH $T \cap T'$ FINITE.

GIVEN A CHAIN $\mathcal{T}$ AND A SET X, LET US SAY

X IS **BELOW** $\mathcal{T}$ IF $X \subseteq^* T$ FOR EACH $T \in \mathcal{T}$,

X IS **COMPATIBLE WITH** $\mathcal{T}$ IF $X \cap T$ IS INFINITE FOR EACH $T \in \mathcal{T}$,

X **MEETS THE BOUNDARY OF** $\mathcal{T}$ IF FOR EACH $T \in \mathcal{T}$ THERE IS SOME $T' \in \mathcal{T}$ WITH $X \cap (T \setminus T')$ INFINITE.

OBSERVATION 18. LET γ BE AN ORDINAL OF COUNTABLE COFINALITY, LET $\mathcal{T} = \{T_\alpha : \alpha < \gamma\}$ BE A CHAIN. THEN THERE IS A FAMILY OF CHAINS

$\{T_\xi : \xi < \mathfrak{b}, \operatorname{cf}(\xi) = \omega\}$ WITH EACH T_ξ OF LENGTH $\gamma + \omega$, $T \subseteq T_\xi$, SUCH THAT FOR EACH $X \in [\omega]^\omega$, IF X MEETS THE BOUNDARY OF T, THEN $|\{\xi < \mathfrak{b} : X$ MEETS THE BOUNDARY OF $T_\xi\}| = \mathfrak{b}$.

PROOF. CHOOSE A SEQUENCE $\gamma_0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n \ldots$ OF TYPE ω, COFINAL IN γ. FOR $n \in \omega$, LET $R_n = \bigcap_{i<n} T_{\gamma_i} \setminus T_{\gamma_n}$. THE FAMILY $\{R_n : n < \omega\}$ IS PAIRWISE DISJOINT AND CONSISTS OF INFINITE SETS.

LET $R_n = \{r(n,k) : k \in \omega\}$.

FOR A MAPPING $f : \omega \to \omega$, PUT $B(f) = \{r(n,k) : n \in \omega, k < f(n)\}$.

CHOOSE A FAMILY $\{f_\xi : \xi < \mathfrak{b}\} \subseteq {}^\omega\omega$ WITH NO UPPER BOUND IN THE ORDER $\leq^*$, AND SUCH THAT FOR EACH $\xi < \eta < \mathfrak{b}$, $B(f_\eta) \setminus B(f_\xi)$ IS INFINITE, AND EACH f_ξ IS INCREASING.

FOR EACH $\xi < \mathfrak{b}$ OF COUNTABLE COFINALITY, SELECT A STRICTLY INCREASING SEQUENCE $\langle \xi_i : i \in \omega \rangle$ CONVERGING TO ξ AND DEFINE

$$\mathcal{T}_\xi = \mathcal{T} \cup \{B(f_\xi) \setminus \bigcup_{i<j} B(f_{\xi_j}) : j \in \omega\}$$

CLEARLY, EACH $\mathcal{T}_\xi$ IS A CHAIN OF LENGTH $\gamma + \omega$.

SUPPOSE $X \subseteq \omega$ MEETS THE BOUNDARY OF $\mathcal{T}$. HENCE X HAS AN INFINITE INTERSECTION WITH INFINITELY MANY R_n'S. THE SAME IS TRUE FOR THE

SET $X \setminus B(f)$, WITH AN ARBITRARY $f \in {}^{\omega}\omega$, BECAUSE FOR EACH f AND FOR EACH $n \in \omega$, $R_n \cap B(f)$ IS FINITE.

LET $\eta < \mathfrak{b}$ BE ARBITRARY. FIND $\xi_0 > \eta$ SUCH THAT $(X \setminus B(f_\eta)) \cap B(f_{\xi_0})$ IS INFINITE. INDUCTION: KNOWING ξ_i, CHOOSE ξ_{i+1} SUCH THAT THE SET $(X \setminus B(f_{\xi_i})) \cap B(f_{\xi_{i+1}})$ IS INFINITE. SINCE $\mathfrak{b}$ IS REGULAR UNCOUNTABLE, $\sup_{i<\omega} \xi_i = \xi < \mathfrak{b}$ AND THE CONSTRUCTION GUARANTEES THAT X MEETS THE BOUNDARY OF $\mathcal{T}_\xi$. $\square$

THEOREM. THERE EXISTS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY.

PROOF. CONSIDER A TREE OF HEIGHT ω_1, CONSISTING OF ALL MAPPINGS $s:\alpha\to\mathfrak{c}$ WITH ALL VALUES $s(\beta)$ LIMIT ORDINALS OF COUNTABLE COFINALITY; CALL THIS TREE Γ.

FOR EACH $s\in\Gamma$, WE SHALL FIND A TOWER $\mathcal{T}_s$ OF COUNTABLE LENGTH AND A SET A_s, WHICH IS BELOW $\mathcal{T}_s$. THE FAMILY $\{A_s : s\in\Gamma\}$ WILL BE AS REQUIRED.

PROCEED BY TRANSFINITE INDUCTION TO ω_1.

START: LET $\mathcal{T}_\emptyset$ BE AN ARBITRARY TOWER OF LENGTH ω, LET $A_\emptyset$ BE AN ARBITRARY SET BELOW $\mathcal{T}_\emptyset$.

INDUCTION STEP, $\alpha < \omega_1$, $\alpha = \beta + 1$:

WE KNOW ALL TOWERS $\mathcal{T}_s$ FOR $s : \beta \to \mathfrak{b}$, $s \in \Gamma$, AND ALL SETS A_s, EACH A_s BELOW $\mathcal{T}_s$.

GIVEN s, WE HAVE TO DEFINE TOWERS $\mathcal{T}_{s^\frown\xi}$ AND SETS $A_{s^\frown\xi}$ FOR ALL $\xi < \mathfrak{b}$, $cf(\xi) = \omega$. SINCE A_s IS BELOW $\mathcal{T}_s$, WE SHALL APPLY OBSERVATION 18, CHOOSING THE UNBOUNDED FAMILY $\{f_\xi : \xi < \mathfrak{b}\}$ IN SUCH A WAY THAT $A_s \subseteq B(f_0)$. KNOWING ALL $\mathcal{T}_s$ FOR $s : \alpha \to \mathfrak{b}$, $s \in \Gamma$, CHOOSE A_s INFINITE AND BELOW $\mathcal{T}_s$ ARBITRARILY.

INDUCTION STEP, $\alpha<\omega_1$, α LIMIT:

KNOWING ALL $\mathcal{T}_s$ AND A_s FOR ALL $s\in\Gamma$ WITH $dom(s)\subset\alpha$, DEFINE FOR $s:\alpha\to b$, $s\in\Gamma$, THE TOWER $\mathcal{T}_s$ SIMPLY AS $\bigcup_{\beta<\alpha}\mathcal{T}_{s\restriction\beta}$.

IT REMAINS TO FIND SETS A_s. DENOTE BY $\mathfrak{X}$ THE FAMILY OF ALL $X\subseteq\omega$ SUCH THAT THE FAMILY

$$\{\mathcal{T}_s : s\in\Gamma,\ dom(s)=\alpha,\ \text{THE SET } X \text{ MEETS THE BOUNDARY OF } \mathcal{T}_s\}$$

IS OF SIZE 2^ω.

ASSIGN TO EACH $X\in\mathfrak{X}$ ONE $s:\alpha\to b$, $s\in\Gamma$, SUCH THAT X MEETS THE BOUNDARY OF $\mathcal{T}_s$ IN A ONE-TO-ONE WAY, DENOTE THIS s AS $s(X)$.

THEN LET A_s BE AN ARBITRARY INFINITE SET BELOW $\mathcal{T}_s$ FOR ALL $s \notin \{s(X) : X \in \mathcal{X}\}$, FOR $X \in \mathcal{X}$ WE DEMAND MOREOVER $A_{s(X)} \subseteq X$.

THIS WORKS.

THE FAMILY $\{A_s : s \in \Gamma\}$ IS ALMOST DISJOINT:

SUPPOSE $s, t \in \Gamma$, $s \subset t$:

DENOTE $\alpha = dom(s)$. BY INDUCTION STEP $\alpha \mapsto \alpha + 1$, WE CHOOSE FOR EXTENDING $\mathcal{T}_s$ A FAMILY $\{f_\xi : \xi < \mathfrak{b}\}$ WITH $A_s \subseteq B(f_0)$. THE CONSTRUCTION IN OBSERVATION 18 GUARANTEES THAT EACH TOWER $\mathcal{T}_{s^\frown \xi}$ CONTAINS SOME MEMBER DISJOINT WITH $B(f_0)$ SO $A_s \subseteq B(f_0)$, $A_t \cap B(f_0)$ IS FINITE.

SUPPOSE $s, t \in \Gamma$ ARE INCOMPARABLE. CHOOSE MINIMAL $\alpha < \omega_1$ SATISFYING $s \restriction \alpha \neq t \restriction \alpha$. THIS α MUST BE A SUCCESSOR ORDINAL, $\alpha = \beta + 1$. DENOTE BY $p = s \restriction \beta = t \restriction \beta$. THEN $\mathcal{T}_s \supseteq \mathcal{T}_{p^\frown s(\beta)}$ AND $\mathcal{T}_t \supseteq \mathcal{T}_{p^\frown t(\beta)}$. ASSUME $s(\beta) < t(\beta)$. THEN THE SET $A_s \subseteq^* B(f_{s(\beta)})$ AND $A_t \subseteq^* B(f_{t(\beta)}) \setminus B(f_{s(\beta)})$.

THE FAMILY $\{A_s : s \in \Gamma\}$ IS COMPLETELY SEPARABLE: SUPPOSE $X \subseteq \omega$, $\{s \in \Gamma : |X \cap A_s| = \omega\}$ IS INFINITE. THEN THERE IS SOME $t \in \Gamma$ SUCH THAT X MEETS THE BOUNDARY OF $\mathcal{T}_t$

TO SEE THIS, LET $\alpha < \omega_1$ BE THE SMALLEST ONE WITH THE SET $K = \{s \in \Gamma : |X \cap A_s| = \omega \ \&\ dom(s) \subseteq \alpha\}$ INFINITE.

IF $\alpha = \beta + 1$, THEN FOR SOME $t \in \Gamma$, $dom(t) = \beta$, $\{s \in K : t \subset s\}$ IS INFINITE. BY THE CONSTRUCTION, THERE IS AN INCREASING SEQUENCE OF LIMIT ORDINALS IN b, $\langle \xi_n : n \in \omega \rangle$ WITH EACH $t^\frown \xi_n \in K$. THE UNBOUNDED FAMILY OF MAPPINGS IN ${}^\omega\omega$ WAS $\{f_\xi : \xi < b\}$; THE SET $B(f_{\xi_n}) \setminus B(f_{\xi_{n-1}}) \in \mathcal{T}_{t^\frown \xi_n}$ AND $A_{t^\frown \xi_n}$ IS BELOW $\mathcal{T}_{t^\frown \xi_n}$. THEFORE FOR $\xi = \sup_{n \in \omega} \xi_n$ WE HAVE THAT X MEETS THE BOUNDARY OF $\mathcal{T}_{t^\frown \xi}$,

SINCE FOR EACH n, $|X \cap A_{t^\frown \xi_n}| = \omega$.

IF α IS A LIMIT ORDINAL, CONSIDER THE SET $\{t \in \Gamma : \text{FOR SOME } s \in K,\ A_s \text{ IS BELOW } \mathcal{T}_t\} = T$.

SINCE T IS COUNTABLE TREE AND ALL LEVELS OF T EXCEPT POSSIBLY THE LAST ONE ARE FINITE, THERE IS A COFINAL BRANCH IN T.

CHOOSING $t_0 \subset t_1 \subset \ldots \subset t_n \subset \ldots$ IN T, COFINAL PART OF THIS BRANCH, THEN THE TOWER $\mathcal{T}_s \in \Gamma$ FOR $s = \bigcup_{n \in \omega} t_n$ SATISFIES THAT X MEETS THE BOUNDARY OF $\mathcal{T}_s$.

WE HAVE VERIFIED THAT FOR $X \in \mathcal{J}^+(\{A_s : s \in \Gamma\})$ THERE IS SOME $s \in \Gamma$ SUCH THAT X MEETS

MEETS THE BOUNDARY OF $\mathcal{T}_s$.

SUPPOSE $\mathrm{dom}(s) = \alpha$. BY OBSERVATION 18, THERE ARE $\mathfrak{b}$-MANY $t \supseteq s$ WITH $\mathrm{dom}(t) = \alpha + 1$ SUCH THAT X MEETS THE BOUNDARY OF $\mathcal{T}_t$, AND FOR EACH SUCH t THERE ARE $\mathfrak{b}$-MANY $p \supseteq t$ WITH $\mathrm{dom}(p) = \alpha + 2$ SUCH ...

SO WHEN CONSTRUCTING ALL A_s WITH $\mathrm{dom}(s) = \alpha + \omega$, WE HAVE $X \in \mathcal{X}$ IN THIS STEP OF RECURSION. SO FOR SOME s WITH $\mathrm{dom}(s) = \alpha + \omega$ WE HAVE GUARANTEED THAT $A_s \subseteq X$. $\square$

COROLLARY. LET $\mathcal{C}$ BE A COUNTABLE ALMOST DISJOINT FAMILY, LET $\mathcal{D}$ BE A DENSE SUBSET OF $\mathcal{P}(\omega)/fin$. THEN THERE IS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY $\mathcal{A} \supseteq \mathcal{C}$, SUCH THAT FOR EACH $A \in \mathcal{A} \setminus \mathcal{C}$, $A^* \in \mathcal{D}$. MOREOVER, $\mathcal{A}$ REFINES $\mathcal{I}^+(\mathcal{C})$. $\square$

THEOREM. ASSUME $s = \omega_1$. THEN RPC(ω) HOLDS TRUE.

PROOF. LET $\mathcal{B}$ BE AN ARBITRARY MAD FAMILY ON ω. FIX A SPLITTING FAMILY $\{Q_\alpha : \alpha < \omega_1\}$; DENOTE $Q_\alpha(0) = Q_\alpha$, $Q_\alpha(1) = \omega \setminus Q_\alpha$. FOR EACH $\alpha < \omega_1$ AND EACH MAPPING $s : \alpha \to 2$, THE FILTER $\mathcal{F}_s$, GENERATED BY $\{Q_\beta(s(\beta)) : \beta < \alpha\}$ HAS A

COUNTABLE BASIS, SO THERE IS A TOWER $\mathcal{T}_s$, WHICH IS A BASIS OF $\mathcal{F}_s$.

FOR EACH $\alpha < \omega_1$ AND FOR EACH $s : \alpha \to 2$, THERE IS A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY $\mathcal{D}_s$, CONSISTING OF SETS BELOW $\mathcal{T}_s$ AND REFINING ALL SETS, WHICH MEET THE BOUNDARY OF $\mathcal{T}_s$. PUT $\mathcal{D}_\alpha = \bigcup \{ \mathcal{D}_s : s \in {}^\alpha 2 \}$.

THE FAMILY $\mathcal{D}_\alpha$ IS COMPLETELY SEPARABLE. WE CAN ALSO CHOOSE $\mathcal{D}_\alpha$ IN SUCH A WAY THAT FOR EACH $D \in \mathcal{D}_\alpha$ THERE IS A UNIQUE $B \in \mathcal{B}$ SATISFYING $D \subseteq B$.

PUT $\mathcal{A}_0 = \mathcal{D}_0$ AND FOR $\alpha < \omega_1$, LET $\mathcal{A}_\alpha = \{ D \in \mathcal{D}_\alpha : (\forall \beta < \alpha)(\forall A \in \mathcal{A}_\beta)$ $D \cap A$ IS FINITE. $\}$

THE FAMILY $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ IS THE REQUIRED ALMOST DISJOINT REFINEMENT OF $\mathcal{J}^+(\mathcal{B})$. OBVIOUSLY, $\mathcal{A}$ IS ALMOST DISJOINT. SO IF $M \in \mathcal{J}^+(\mathcal{B})$ WE HAVE TO FIND SOME $A \in \mathcal{A}$ WITH $A \subseteq M$.

HOWEVER, WE NEED LESS. IT IS ENOUGH TO PROVE THAT FOR SOME $\alpha < \omega_1$, $M \in \mathcal{J}^+(\mathcal{A}_\alpha)$.

TO SEE THIS, SUPPOSE THAT FOR SOME α, $M \in \mathcal{J}^+(\mathcal{A}_\alpha)$. CHOOSE THE α TO BE THE SMALLEST ONE.

IF $\alpha = 0$, THEN $\mathcal{A}_\alpha = \mathcal{D}_\alpha$ AND $\mathcal{D}_\alpha$ IS COMPLETELY SEPARABLE, HENCE M CONTAINS SOME $D \in \mathcal{D}_0 = \mathcal{A}_0$.

IF $\alpha > 0$, THEN FOR EACH $\beta < \alpha$,

THE SET $\{A \in \mathcal{A}_\beta : |A \cap M| = \omega\}$ IS FINITE.

THERE IS AN INFINITE COUNTABLE SUBSET $\mathcal{A}' \subseteq \mathcal{A}_\alpha$ SUCH THAT FOR EACH $A' \in \mathcal{A}'$, $|A' \cap M| = \omega$ AND BY THE CHOICE OF $\mathcal{A}_\alpha$, $A' \cap A$ IS FINITE FOR EACH $A \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta$.

CONSEQUENTLY, THERE IS A SUBSET $M' \subseteq M$, SUCH THAT M' IS ALMOST DISJOINT WITH ALL $A \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta$

AND $M' \cap A'$ IS INFINITE FOR EACH $M' \in \mathcal{A}'$. HENCE $M' \in \mathcal{J}^+(\mathcal{D}_\alpha)$, SO FOR SOME $D \in \mathcal{D}_\alpha$, $D \subseteq M'$.

NOW D MUST BELONG TO $\mathcal{A}_\alpha$, SINCE IT IS ALMOST DISJOINT WITH ALL $A \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta$

SO FIX AN $M \in \mathcal{J}^+(\mathcal{B})$ AND WE HAVE TO SHOW THAT FOR SOME α, $M \in \mathcal{J}^+(\mathcal{A}_\alpha)$. CHOOSE $\alpha_0 < \omega_1$ SUCH THAT FOR SOME $s_0 : \alpha_0 \to 2$ WE HAVE THAT M MEETS THE BOUNDARY OF $\mathcal{T}_{s_0}$ AND THAT FOR EACH $T \in \mathcal{T}_{s_0}$, $M \cap T \in \mathcal{J}^+(\mathcal{B})$.

IF $M \in \mathcal{J}^+(\mathcal{A}_\beta)$, FOR SOME $\beta \leq \alpha_0$, WE ARE DONE. BUT IF $M \notin \mathcal{J}^+(\mathcal{A}_\beta)$ FOR ALL $\beta \leq \alpha_0$, THEN THERE IS A COUNTABLE SUBSET $\mathcal{B}_0 \subseteq \mathcal{B}$ SUCH THAT FOR EACH $\beta \leq \alpha_0$ AND FOR EACH $A \in \mathcal{A}_\beta$, IF $|A \cap M| = \omega$, THEN $A \subseteq B$ FOR SOME $B \in \mathcal{B}_0$.

THERE IS A SET $M_0' \subseteq M$ WITH THE FOLLOWING PROPERTIES:

a) $M_0' \in \mathcal{J}^+(\mathcal{B})$;

b) $M_0' \cap B$ IS FINITE FOR EACH $B \in \mathcal{B}_0$

c) $M_0' \cap T$ IS INFINITE FOR EACH $T \in \mathcal{T}_{\beta_0}$.

OBSERVE: THE SET M_0' DOES NOT MEET THE BOUNDARY OF $\mathcal{T}_{\beta_0}$ — IN THE OPPOSITE CASE THE WOULD EXIST SOME $D \in \mathcal{D}_{\alpha_0}$, $D \subseteq M_0'$ AND SINCE $D \subseteq B$ FOR NO $B \in \mathcal{B}_0$, $D \in \mathcal{A}_{\alpha_0}$, CONTRARY TO OUR ASSUMPTION.

SO THERE IS SOME $T \in \mathcal{T}_{\beta_0}$ SUCH THAT $M_0' \cap T$ IS BELOW $\mathcal{T}_{\beta_0}$. CHOOSE $B_0 \in \mathcal{B}$ SUCH THAT $M_0' \cap T \cap B_0$ IS INFINITE AND PUT

$M_0 = M_0' \cap T \setminus B_0$.

LET US REPEAT THE PREVIOUS TRY WITH THE SET M_0: LET $\alpha_1 > \alpha_0$ BE SUCH THAT FOR SOME $s_1 > s_0$, $s_1 : \alpha_1 \to 2$ WE HAVE THAT M_0 MEETS THE BOUNDARY OF $\mathcal{T}_{s_1}$ AND FOR EACH $T \in \mathcal{T}_{s_1}$, $M_0 \cap T \in J^+(\mathcal{B})$.

ASSUME THAT WE WERE UNLUCKY AGAIN: FOR EACH $\beta \leq \alpha_1$, $M_0 \notin J^+(\mathcal{A}_\beta)$. APPLY THE PREVIOUS REASONING TO GET COUNTABLE $\mathcal{B}_1 \subseteq \mathcal{B}$, $B_1 \in \mathcal{B}$ AND $M_1 \subseteq M_0$, $M_1 \in J^+(\mathcal{B})$.

CONTINUE: $\alpha_2 > \alpha_1$, B_2, $\mathcal{B}_2$, M_2 ... ETC. IF WE NEEDED ALL ω-MANY STEPS, THEN WE GET $\tilde{\alpha} = \sup_{n \in \omega} \alpha_n$ AND $s = \bigcup_{n \in \omega} s_n$.

SINCE $\mathcal{P}(\omega)/\mathrm{fin}$ HAS A STRONG COUNTABLE SEPARATION PROPERTY, THERE IS A SET $L \subseteq M$ SUCH THAT $L \supseteq^* M \cap B_n$ FOR ALL $n \in \omega$ AND $|L \cap B| < \omega$ FOR EACH $B \in \bigcup_{n \in \omega} \mathcal{B}_n$.

CLEARLY, THE SET $L \in \mathcal{J}^+(\mathcal{B})$ AND L MEET THE BOUNDARY OF $\mathcal{T}_s$. SO $L \in \mathcal{J}^+(\mathcal{D}_\mathcal{Z})$.

IF $D \in \mathcal{D}_\mathcal{Z}$ SATISFIES $D \subset L$, THE D IS ALMOST DISJOINT WITH EACH $B \in \bigcup_{n \in \omega} \mathcal{B}_n$, SO $D \in \mathcal{A}_\mathcal{Z}$.

THUS $L \in \mathcal{J}^+(\mathcal{A}_\mathcal{Z})$, SO $M \in \mathcal{J}^+(\mathcal{A}_\mathcal{Z})$ AS WELL. $\square$

PROBLEM. IS THERE A COMPLETELY SEPARABLE MAD FAMILY ?

THEOREM. THE FOLLOWING ARE EQUIVALENT:

(i) $\mathrm{RPC}(\omega)$;

(ii) FOR EACH INFINITE MAD FAMILY $\mathcal{B}$ ON ω, $\mathcal{J}^{+}(\mathcal{B})$ HAS AN ALMOST DISJOINT REFINEMENT BY A COMPLETELY SEPARABLE ALMOST DISJOINT FAMILY;

(iii) THERE IS A BASE TREE T SUCH THAT EVERY MAD FAMILY $\mathcal{A} \subseteq T$ IS COMPLETELY SEPARABLE.

PROOF. (iii) $\to$ (ii) $\to$ (i) IS TRIVIAL.

(i) $\to$ (iii): LET $\mathcal{H} = \{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ BE AN ARBITRARY BASE MATRIX. PROCEEDING BY A TRANSFINITE

INDUCTION, FIND A NEW BASE MATRIX $\{\mathcal{B}_\alpha : \alpha < \mathfrak{h}\}$ SUCH THAT

a) FOR EACH $\alpha < \mathfrak{h}$, $\mathcal{B}_\alpha \prec \mathcal{A}_\alpha$;

b) $|\mathcal{B}_0| = 2^\omega$;

c) FOR EACH $\alpha < \mathfrak{h}$, $\mathcal{B}_{\alpha+1}$ IS AN ALMOST DISJOINT REFINEMENT OF $\mathcal{J}^+(\mathcal{B}_\alpha)$ - HERE (i) IS USED.

FOR EACH $\alpha < \mathfrak{h}$ AND FOR EACH $B \in \mathcal{B}_\alpha$ CHOOSE A UNIQUE $C(B) \in \mathcal{B}_{\alpha+1}$ SATISFYING $C(B) \subseteq B$.

THE FAMILY $\{C(B) : B \in \bigcup_{\alpha<\mathfrak{h}} \mathcal{B}_\alpha\}$ IS DENSE IN $(\mathcal{P}(\omega), \subseteq^*)$ AND IS A TREE UNDER $\subseteq^*$.

LET $\mathcal{D} \subseteq \{C(B) : B \in \bigcup_{\alpha<\mathfrak{h}} \mathcal{B}_\alpha\}$ BE AN ARBITRARY MAD FAMILY AND

LET $M \in \mathcal{J}^+(\mathcal{D})$ BE ARBITRARY.

FOR EACH $\alpha < \mathfrak{h}$ LET

$\mathcal{B}_\alpha(M) = \{B \in \mathcal{B}_\alpha \cap \mathcal{D} : |B \cap M| = \omega\}$.

LET $\alpha < \mathfrak{h}$ BE THE FIRST ONE WITH

$\bigcup_{\beta < \alpha} \mathcal{B}_\beta(M)$ INFINITE.

CASE 1: $\alpha = \beta + 1$. WE HAVE $\mathcal{B}_\beta(M)$ INFINITE. THE COLLECTION $\bigcup_{\gamma < \beta} \mathcal{B}_\gamma(M)$ IS FINITE AND $\mathcal{D}$ IS ALMOST DISJOINT, HENCE EACH $B \in \mathcal{B}_\beta(M)$ HAS AN INFINITE INTERSECTION ALSO WITH $M' = M \setminus \bigcup_{\gamma < \beta} \bigcup \mathcal{B}_\gamma(M)$. SINCE $\mathcal{D} \subseteq \{C(B) : B \in \bigcup_{\alpha < \mathfrak{h}} \mathcal{B}_\alpha\}$, WE GET THAT $M' \in \mathcal{J}^+(\mathcal{B}_{\beta - 1})$. BY c), THERE IS SOME $B \in \mathcal{B}_\beta$ WITH $B \subseteq M'$.

SINCE B IS DISJOINT WITH $\bigcup_{\gamma<\beta} \mathcal{B}_\gamma(M)$ AND SINCE $\mathcal{D}$ IS MAXIMAL, THERE MUST BE SOME $D \in \mathcal{D}$ WITH $D \subseteq^* M'$.

CASE 2: α IS A LIMIT ORDINAL. CONSIDER A FAMILY $\{B \in \mathcal{B}_\alpha :$ FOR EACH $\beta<\alpha$ AND EACH $B' \in \mathcal{B}_\beta(M)$, $B \cap B'$ IS FINITE$\} := \mathcal{B}'_\alpha$. THERE IS A SET $M' \subseteq M$ SUCH THAT $M' \in \mathcal{J}^+(\mathcal{B}'_\alpha)$ AND $M' \cap B$ IS FINITE FOR EACH $B \in \bigcup_{\beta<\alpha} \mathcal{B}_\beta(M)$. BY c), THERE IS SOME $B \in \mathcal{B}_{\beta+1}$ WITH $B \subseteq M'$ AND WE MAY CONTINUE AS IN CASE 1. $\square$

OBSERVATION 19. SUPPOSE THAT EACH DENSE SUBSET OF $([\omega]^{\omega}, \subseteq^{*})$ CONTAINS A COMPLETELY SEPARABLE MAD FAMILY. THEN RPC(ω) HOLDS TRUE.

— INDEED, FOR A MAD FAMILY $\mathcal{Q}$ CONSIDER $\mathcal{D} = \{D \in [\omega]^{\omega}$: FOR SOME $Q \in \mathcal{Q}$, $D \subseteq Q\}$. ANY COMPLETELY SEPARABLE MAD FAMILY $\mathcal{A} \subseteq \mathcal{D}$ IS THE ALMOST DISJOINT REFINEMENT OF $\mathcal{J}^{+}(\mathcal{Q})$.

SAHARON SHELAH PROVED IN 2009 THE STRONGEST KNOWN STATEMENT ABOUT RPC(ω).